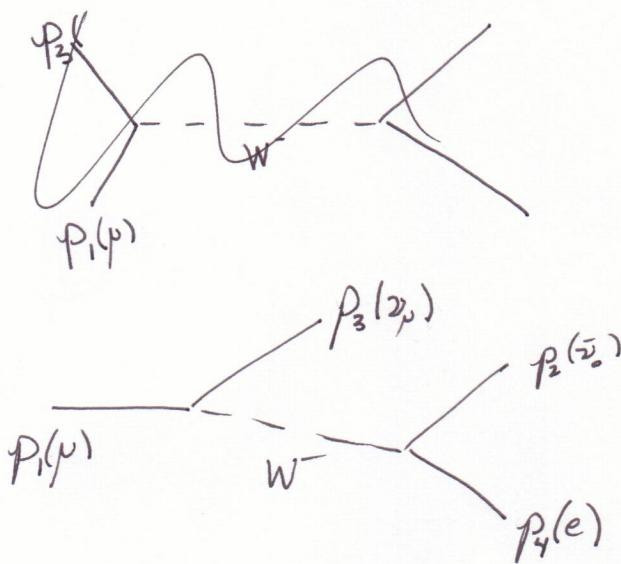


Muon Decay

$$\mu \rightarrow e + \nu_\mu + \bar{\nu}_e$$



Let's calculate the decay rate beginning with the invariant amplitude (squared and spin-averaged) given.

$$M = \frac{g_w^2}{8m_w^2} [\bar{u}(3)\gamma^\mu(1-\gamma^5)u(1)] [\bar{u}(4)\gamma_\mu(1-\gamma^5)v(2)]$$

$$\langle |M|^2 \rangle = \frac{2g_w^4}{m_w^4} (\vec{p}_1 \cdot \vec{p}_2)(\vec{p}_3 \cdot \vec{p}_4)$$

Like usual, view the decay from the muon's rest frame. Also, let the masses of the neutrinos and electron be zero since they're so small.

$$\vec{p}_1(\mu) \cdot \vec{p}_2(\bar{\nu}) = (m_\mu, \vec{0}) \cdot (E_{\bar{\nu}}, \vec{p}_{\bar{\nu}}) = m_\mu E_{\bar{\nu}}$$

Also, by conservation of 4-momentum

$$\vec{p}_1(\mu) = \vec{p}_2(\bar{\nu}) + \vec{p}_3(\nu) + \vec{p}_4(e)$$

$$\vec{p}_1(\mu) - \vec{p}_2(\bar{\nu}) = \vec{p}_3(\nu) + \vec{p}_4(e)$$

$$(\varphi_1(\mu) - \varphi_2(\bar{\nu}))^2 = (\varphi_3(\nu) + \varphi_4(e))^2$$

$$\varphi_1^2(\mu) + \varphi_2^2(\bar{\nu}) - 2\varphi_1(\mu) \cdot \varphi_2(\bar{\nu}) = \varphi_3^2(\nu) + \varphi_4^2(e) + 2\varphi_3(\nu) \cdot \varphi_4(e)$$

$$(m_\mu \vec{0})^2 + (E_{\bar{\nu}}, \vec{\varphi}_2(\bar{\nu}))^2 - 2\varphi_1(\mu) \cdot \varphi_2(\bar{\nu}) = (E_\nu, \vec{\varphi}_3(\nu))^2 + (E_e, \varphi_4(e))^2 + 2\varphi_3(\nu) \cdot \varphi_4(e)$$

$$m_\mu^2 + (E_{\bar{\nu}}^2 - |\vec{\varphi}_2(\bar{\nu})|^2) - 2m_\mu E_{\bar{\nu}} = (E_\nu^2 - |\vec{\varphi}_3(\nu)|^2) + (E_e^2 - |\vec{\varphi}_4(e)|^2) + 2\varphi_3(\nu) \cdot \varphi_4(e)$$

$$m_\mu^2 + m_\nu^2 - 2m_\mu E_{\bar{\nu}} = m_3^2 + m_4^2 + 2\varphi_3 \cdot \varphi_4$$

$$\frac{1}{2} m_\mu^2 - m_\mu E_{\bar{\nu}} = \varphi_3 \cdot \varphi_4$$

Rewrite the squared amplitude

$$\langle |M|^2 \rangle = \frac{g_w^4}{m_w^4} (m_\mu E_{\bar{\nu}})(m_\mu (m_\mu - E_{\bar{\nu}}))$$

$$\langle |M|^2 \rangle = \frac{g_w^4}{m_w^4} m_\mu^2 E_{\bar{\nu}} (m_\mu - 2E_{\bar{\nu}})$$

On to the decay rate

$$d\Gamma = \frac{1}{2m_\mu} \left(\frac{d^3 p_2}{(2\pi)^3 2E_2} \frac{d^3 p_3}{(2\pi)^3 2E_3} \frac{d^3 p_4}{(2\pi)^3 2E_4} \right) \langle |M|^2 \rangle \delta^4(p_1 - p_2 - p_3 - p_4)$$

$$d\Gamma = \frac{1}{2m_\mu} \left(\frac{d^3 p_2}{(2\pi)^3 2E_2} \frac{d^3 p_3}{(2\pi)^3 2E_3} \frac{d^3 p_4}{(2\pi)^3 2E_4} \right) \langle |M|^2 \rangle \delta(m_\mu - E_2 - E_3 - E_4) \delta^{(2)}(-\vec{p}_2 - \vec{p}_3 - \vec{p}_4)$$

$$\Gamma = \frac{1}{2m_p \cdot 8(2\pi)^5} \int d^3 p_2 d^3 p_3 d^3 p_4 \frac{\langle |M|^2 \rangle}{E_2 E_3 E_4} \delta(m_p - E_2 - E_3 - E_4) \delta^{(3)}(-\vec{p}_2 - \vec{p}_3 - \vec{p}_4)$$

This happens to be a situation where all three products can be approximated as massless. Hence, $E = |\vec{p}|$.

$$\Gamma = \frac{1}{16m_p (2\pi)^5} \int d^3 p_2 d^3 p_3 d^3 p_4 \frac{\langle |M|^2 \rangle}{|\vec{p}_2| |\vec{p}_3| |\vec{p}_4|} \delta(m_p - E_2 - E_3 - E_4) \delta^{(3)}(-\vec{p}_2 - \vec{p}_3 - \vec{p}_4)$$

Perform \vec{p}_3 integration,

$$\Gamma = \frac{1}{16m_p (2\pi)^5} \int d^3 p_2 \cancel{d^3 p_3} d^3 p_4 \frac{\langle |M|^2 \rangle}{|\vec{p}_2| |\vec{p}_2 + \vec{p}_4| |\vec{p}_4|} \delta(m_p - |\vec{p}_2| - |\vec{p}_3| - |\vec{p}_4|)$$

Perform \vec{p}_2 integration. Change to spherical coordinates with $d^3 p_2 = |\vec{p}_2|^2 d|\vec{p}_2| \sin\theta d\theta d\phi$.

$$\Gamma = \frac{1}{16m_p (2\pi)^5} \int d^3 p_4 d|\vec{p}_2| \frac{|\vec{p}_2| \sin\theta d\theta d\phi \langle |M|^2 \rangle}{|\vec{p}_2 + \vec{p}_4| |\vec{p}_4|} \delta(m_p - |\vec{p}_2| - |\vec{p}_3| - |\vec{p}_4|)$$

We still have an issue in dealing with the denominator.

$$\begin{aligned} |\vec{p}_2 + \vec{p}_4| &= \sqrt{|\vec{p}_2 + \vec{p}_4|^2} \\ &= \sqrt{|\vec{p}_2|^2 + |\vec{p}_4|^2 + 2 \vec{p}_2 \cdot \vec{p}_4} \\ &= \sqrt{|\vec{p}_2|^2 + |\vec{p}_4|^2 + 2 |\vec{p}_2| |\vec{p}_4| \cos\theta} \end{aligned}$$

$$\Gamma = \frac{2\pi}{16m_p(2\pi)^5} \int d\vec{p}_4^3 d|\vec{p}_2| \frac{|\vec{p}_2| \sin\theta d\theta \langle |M|^2 \rangle}{|\vec{p}_4| \sqrt{|\vec{p}_2|^2 + |\vec{p}_4|^2 + 2|\vec{p}_2||\vec{p}_4|\cos\theta}} \delta(m_p - |\vec{p}_2| - |\vec{p}_3| - |\vec{p}_4|)$$

$$\text{Let } x = \sqrt{|\vec{p}_2|^2 + |\vec{p}_4|^2 + 2|\vec{p}_2||\vec{p}_4|\cos\theta} = E_3$$

We're changing integration variables from θ to x .

$$dx = \frac{1}{2} \left(|\vec{p}_2|^2 + |\vec{p}_4|^2 + 2|\vec{p}_2||\vec{p}_4|\cos\theta \right)^{-\frac{1}{2}} 2|\vec{p}_2||\vec{p}_4|\sin\theta d\theta$$

$$\frac{dx}{|\vec{p}_4|} = \frac{|\vec{p}_2| \sin\theta d\theta}{\sqrt{|\vec{p}_2|^2 + |\vec{p}_4|^2 + 2|\vec{p}_2||\vec{p}_4|\cos\theta}}$$

$$\Gamma = \frac{1}{16m_p(2\pi)^4} \int d\vec{p}_4^3 d|\vec{p}_2| \int_{x_-}^{x_+} \frac{dx}{|\vec{p}_4|^2} \langle |M|^2 \rangle \delta(m_p - |\vec{p}_2| - x - |\vec{p}_4|)$$

$$\text{where } x_{\pm} = |E_2 \pm E_4|$$

Remember that x is replacing an angle θ , so the rest of the calculation implicitly depends on respecting the limits of the x -integral. Most especially, the argument of the δ -function requires

$$x_- < x < x_+$$

$$|E_2 - E_4| < (m_p - |\vec{p}_2| - |\vec{p}_4|) < |E_2 + E_4|$$

$$|E_2 - E_4| < (m_p - E_2 - E_4) < E_2 + E_4$$

$$\frac{1}{2} [|E_2 - E_4| + E_2 + E_4] < \frac{1}{2} m_p < E_2 + E_4$$

The term on the left is just the larger of E_2 and E_4 .

$$\max\{E_2, E_4\} < \frac{1}{2}m_p < E_2 + E_4$$

So now we have not two, but three inequality conditions.

$$\begin{cases} E_2 < \frac{1}{2}m_p \\ E_4 < \frac{1}{2}m_p \\ E_2 + E_4 > \frac{1}{2}m_p \end{cases} \quad (E = |\vec{p}|)$$

Keeping these in mind, carry out the α -integral.

$$\Gamma = \frac{1}{(4\pi)^4 m_p} \int d^3 \vec{p}_4 \, d|\vec{p}_2| \frac{\langle |M|^2 \rangle}{|\vec{p}_4|^2}$$

$$\Gamma = \frac{m_p g_w^4}{(4\pi)^4 m_w^4} \int \frac{d^3 \vec{p}_4}{|\vec{p}_4|^2} \int_{\frac{1}{2}m_p - |\vec{p}_4|}^{\frac{1}{2}m_p} d|\vec{p}_2| E_2 (m_p - 2E_2)$$

$$\Gamma = \frac{m_p g_w^4}{(4\pi)^4 m_w^4} \int \frac{d^3 \vec{p}_4}{|\vec{p}_4|^2} \int_{\frac{1}{2}m_p - |\vec{p}_4|}^{\frac{1}{2}m_p} d|\vec{p}_2| |\vec{p}_2| (m_p - 2|\vec{p}_2|)$$

$$\Gamma = \frac{m_p g_w^4}{(4\pi)^4 m_w^4} \int \frac{d^3 \vec{p}_4}{|\vec{p}_4|^2} \left[\frac{m_p}{2} |\vec{p}_2|^2 - \frac{2}{3} |\vec{p}_2|^3 \right]_{\frac{1}{2}m_p - |\vec{p}_4|}^{\frac{1}{2}m_p}$$

$$\Gamma = \frac{m_p g_w^4}{(4\pi)^4 m_w^4} \int \frac{d^3 \vec{p}_4}{|\vec{p}_4|^2} \left[\frac{1}{2} m_p \left(\frac{1}{4} m_p^2 - \left(\frac{1}{2} m_p - |\vec{p}_4| \right)^2 \right) - \frac{2}{3} \left(\frac{1}{8} m_p^3 - \left(\frac{1}{2} m_p - |\vec{p}_4| \right)^3 \right) \right]$$

$$\frac{1}{2} m_p \left(\frac{1}{4} m_p^2 - \left(\frac{1}{4} m_p^2 - m_p |\vec{p}_4| + |\vec{p}_4|^2 \right) \right) - \frac{2}{3} \left(\frac{1}{8} m_p^3 - \left(\frac{1}{8} m_p^3 - 3 \cdot \frac{1}{4} m_p^2 |\vec{p}_4| + 3 \cdot \frac{1}{2} m_p |\vec{p}_4|^2 - |\vec{p}_4|^3 \right) \right)$$

$$\frac{1}{2} m_p (m_p |\vec{p}_4| - |\vec{p}_4|^2) - \frac{2}{3} \left(\frac{3}{4} m_p^2 |\vec{p}_4| - \frac{3}{2} m_p |\vec{p}_4|^2 + |\vec{p}_4|^3 \right)$$

$$\frac{1}{2} m_p^2 |\vec{p}_4| - \frac{1}{2} m_p |\vec{p}_4|^2 - \frac{1}{2} m_p^2 |\vec{p}_4| + m_p |\vec{p}_4|^2 - \frac{2}{3} |\vec{p}_4|^3$$

$$\frac{1}{2} m_p |\vec{p}_4|^2 - \frac{2}{3} |\vec{p}_4|^3$$

$$\Gamma = \frac{m_p g_W^4}{(4\pi)^4 m_w^4} \int d\vec{p}_4^3 \left(\frac{1}{2} m_p |\vec{p}_4|^2 - \frac{2}{3} |\vec{p}_4|^3 \right)$$

$$\Gamma = \frac{m_p g_W^4}{(4\pi)^4 m_w^4} \int d\vec{p}_4 \left(\frac{1}{2} m_p - \frac{2}{3} |\vec{p}_4| \right)$$

$$\Gamma = \frac{m_p g_W^4}{(4\pi)^4 m_w^4} \int d\vec{p}_4 \left(\frac{1}{2} m_p |\vec{p}_4|^2 - \frac{2}{3} |\vec{p}_4|^3 \right)$$

$$\Gamma = \frac{m_p g_W^4}{(4\pi)^3 m_w^4} \int_0^{\frac{1}{2} m_p} \left(\frac{1}{2} m_p |\vec{p}_4|^2 - \frac{2}{3} |\vec{p}_4|^3 \right)$$

$$\Gamma = \frac{m_p g_W^4}{(4\pi)^3 m_w^4} \left[\frac{1}{2} m_p \left[\frac{|\vec{p}_4|^3}{3} - \frac{2}{3} \left[\frac{|\vec{p}_4|^4}{4} \right] \right] \right]_0^{\frac{1}{2} m_p}$$

$$\Gamma = \frac{m_p g_W^4}{m_w^4 (4\pi)^3} \left[\frac{1}{48} m_p^4 - \frac{1}{96} m_p^4 \right] = \frac{m_p g_W^4}{m_w^4 (4\pi)^3} \frac{1}{96} m_p^4$$

$$\boxed{\Gamma = \frac{m_p g_W^4}{96 (4\pi)^3 m_w^4}}$$